

# Explicit chiral symmetry breaking and the $1/N$ expansion in models with multi-fermion interactions

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Explicit chiral symmetry breaking is a natural feature of many QCD inspired models with multi-quark interactions. To carry out the  $1/N$  expansion of these theories, one integrates over the quark fields. In the process of calculating the one-loop fermion determinant which is not chiral invariant due to quark masses, special care must be taken in order not to alter the original symmetry breaking pattern of the Lagrangian. We show how to do this consistently.

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## I. INTRODUCTION

The purpose of this paper is to resolve a problem that arises in models with chirally symmetric multi-fermion interactions when this symmetry is explicitly broken by the mass term of the fundamental fermion field. These can be, in particular, the corresponding extensions of the  $SU(N)$  Thirring model in two space-time dimensions [1], or the Nambu – Jona-Lasinio like models in four space-time dimensions [2, 3]. To carry out the  $1/N$  expansion of these theories, one should integrate over the fermions, facing the task of calculating the one-loop fermion diagrams to obtain the effective action of the auxiliary bosonic fields. The problem is that the standard procedure leads to a result which does not possess the transformation properties of the original fermion Lagrangian with respect to the action of the continuous chiral group.

The essence of the problem has been already discussed in Ref. [4], where the formal solution has been found by adding a functional  $P$  in the bosonic fields and their derivatives to the real part of the fermion determinant. Here we suggest an alternative method to solve the problem. This new solution leads directly to the result and in addition provides for an interpretation of the correcting functional  $P$  in terms of Feynman diagrams.

## II. THE $1/N$ EXPANSION OF THE $SU(N)$ THIRRING MODEL WITH EXPLICITLY BROKEN CHIRAL SYMMETRY

The  $SU(N)$  Thirring model describes a system of  $N$  Dirac fields  $\psi_{(a)}$  carrying the flavor index  $a = 1, 2, \dots, N$ , which we promptly suppress, in two space-time dimensions with the Lagrangian density

$$\mathcal{L}(\psi, \bar{\psi}) = -\bar{\psi}(i\partial\!\!\!/ + \hat{m})\psi + \frac{g^2}{2} [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2], \quad (1)$$

where  $\bar{\psi} = \psi^\dagger\beta$ ,  $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ . The  $2 \times 2$  antihermitian symmetric matrices  $\alpha^\mu = -i\beta\gamma^\mu$ ,  $\mu = 0, 1$  generate the Clifford algebra:  $\{\alpha^\mu, \alpha^\nu\} = 2\delta_{\mu\nu}$ . The matrix  $\beta$  is imaginary and antisymmetric. We take here the choice  $\beta = -\gamma^0 = \sigma_2$ ,  $\gamma^1 = i\sigma_1$ ,  $\gamma_5 = \sigma_3$ , where  $\sigma_i$  are the Pauli matrices. One has also  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$  with the Minkowski metric tensor  $g^{\mu\nu} = \text{diag}(1, -1)$ .

The Lagrangian possesses a  $U(1)$  chiral symmetry  $\psi_{(a)} \rightarrow e^{i\theta\gamma_5}\psi_{(a)}$  which is explicitly broken by the mass term. One can easily find that under small variations  $\theta \ll 1$ , we have

$$\delta\mathcal{L}(\psi, \bar{\psi}) = -\hat{m}\delta(\bar{\psi}\psi) = -2\theta\hat{m}\bar{\psi}i\gamma_5\psi. \quad (2)$$

Following Ref. [1], one can equivalently introduce auxiliary fields  $\sigma$ ,  $\phi$ , and write

$$\mathcal{L}(\psi, \bar{\psi}, \sigma, \phi) = -\bar{\psi}(i\partial\!\!\!/ + \hat{m} + \sigma + i\gamma_5\phi)\psi - \frac{\sigma^2 + \phi^2}{2g^2}. \quad (3)$$

The Euler – Lagrange equations for the bosonic fields are constraints

$$-\frac{\partial\mathcal{L}}{\partial\sigma} = \bar{\psi}\psi + \frac{\sigma}{g^2} = 0, \quad -\frac{\partial\mathcal{L}}{\partial\phi} = \bar{\psi}i\gamma_5\psi + \frac{\phi}{g^2} = 0, \quad (4)$$

which relate the chiral transformations of fermions and bosons, and hence

$$\delta\sigma = -g^2\delta(\bar{\psi}\psi) = 2\theta\phi, \quad (5)$$

$$\delta\phi = -g^2\delta(\bar{\psi}i\gamma_5\psi) = -2\theta\sigma. \quad (6)$$

Then it follows from (2) and (5) that the infinitesimal symmetry transformation is now

$$\delta\mathcal{L} = \frac{\hat{m}}{g^2}\delta\sigma. \quad (7)$$

Performing the Gaussian integral over the fermions in the functional integral corresponding to (3) (see Eq. (18) below), one finds that the real part of the effective action  $S_{\text{eff}}$  has a different transformation property. Indeed, one obtains in euclidean space

$$\text{Re } S_E^{\text{eff}} = \frac{N}{2}\text{Tr} \ln D_E^\dagger D_E - \int d^2x_E \frac{\sigma^2 + \phi^2}{2g^2} \quad (8)$$

with

$$\delta(\text{Re } S_E^{\text{eff}}) = \hat{m}N \text{Tr} \left( \frac{2\theta\phi}{D_E^\dagger D_E} \right), \quad (9)$$

where the Dirac operator is given by  $D_E = -(i\cancel{\partial} + \hat{m} + \sigma + i\gamma_5\phi)$ . Here we use the following convention: a Lorentz 2-vector  $x^\mu$  is continued as  $x^0 \rightarrow -ix_2^E$ ,  $x^1 \rightarrow x_1^E$ . The euclidean  $\gamma$ -matrices  $\gamma_a^E$  are antihermitian  $\gamma^0 \rightarrow \gamma_2^E = -i\sigma_2$ ,  $\gamma^1 \rightarrow \gamma_1^E = i\sigma_1$ . They satisfy the anticommutation relation  $\{\gamma_a^E, \gamma_b^E\} = -2\delta_{ab}$ . Accordingly  $\cancel{\partial} = \gamma_a^E \partial_a^E$  is a hermitian operator. Furthermore  $\text{Tr} = \int d^2x_E \text{tr}$ , where the last trace is over Dirac indices.

The unsuccessful outcome of Eq. (9) shows that the standard evaluation of the Gaussian functional integral over fermions, when chiral symmetry is explicitly broken, must be corrected, otherwise the result is not consistent with the constraint imposed by the symmetry breaking pattern of the theory at the tree level. In particular, this can be done by the insertion of new counterterms into the effective action (8). The corresponding method has been developed in [4]. However, it was not clear then (a) if these counterterms were compatible with the dynamics of the original multi-fermion system, and (b) if the method could be applied to renormalizable models. In the following we shall find the answers to these questions describing a new way of dealing with the problem.

### III. TADPOLE MECHANISM

It is convenient to divide the Lagrangian (3) into two parts  $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ . The first part

$$\mathcal{L}_1 = -\bar{\psi}(i\cancel{\partial} + \hat{m} + \sigma + i\gamma_5\phi)\psi - \frac{\hat{m}\sigma}{g^2} \quad (10)$$

is invariant under the action of the chiral group. We expect that this property will be still fulfilled for the corresponding part of the effective Lagrangian obtained as a result of integration over the  $\psi$  fields. Our expectation is based on the observation that the dangerous symmetry breaking fermion mass term can be easily subtracted if one considers the tree-level tadpole  $\sigma \rightarrow$  vacuum transition contained in  $\mathcal{L}_1$ , as it is shown in Fig. 1.



FIG. 1: The mass term of the fermion is cancelled due to a contribution generated by the  $\sigma \rightarrow$  vacuum transition.

The second part

$$\mathcal{L}_2 = \frac{\hat{m}\sigma}{g^2} - \frac{\sigma^2 + \phi^2}{2g^2} \quad (11)$$

possesses the necessary transformation property  $\delta\mathcal{L}_2 = \hat{m}\delta\sigma/g^2$  of the Lagrangian (3). Note, that the term  $\hat{m}\sigma/g^2$ , which we subtract in (10) and add in (11), is unambiguously determined by the chiral symmetry restriction  $\delta\mathcal{L}_1 = 0$ . It follows from  $\mathcal{L}_2$  additionally that the auxiliary fields  $\sigma$  and  $\phi$  do not propagate at tree level. They have the  $\delta$ -like Green functions

$$\Delta_{\sigma,\phi}(x-y) = -ig^2\delta(x-y). \quad (12)$$

To integrate the fermions out of the Lagrangian  $\mathcal{L}_1$  and not break its transformation properties with respect to the chiral group, we modify the corresponding functional integral, writing it as

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \left\{ \exp \left( \iint \prod_{i=1}^2 d^2x_i \frac{\delta}{\delta\sigma(x_1)} \Delta_{\sigma}(x_1 - x_2) \frac{\delta}{\delta\Gamma(x_2)} \right) \exp \left( -i \int d^2x [\bar{\psi}(i\not{\partial} + \hat{m} + \sigma + i\gamma_5\phi)\psi + \frac{\hat{m}}{g^2}\Gamma] \right) \right\}_{|\Gamma=0} \quad (13)$$

Obviously, this definition coincides with the standard formula at  $\hat{m} = 0$ . For  $\hat{m} \neq 0$  the action of the first exponent upon the second is to include as basic element in the functional integral the chiral symmetric tree-level result shown in Fig. 1. On the other hand, by first integrating out the fermions, this exponent generates an infinite number of all possible one-loop fermion diagrams attached to the external auxiliary fields, including those that contain the  $\sigma \rightarrow$  vacuum transitions, rendering the result invariant under chiral transformations. Some of these diagrams are shown in Fig. 2.

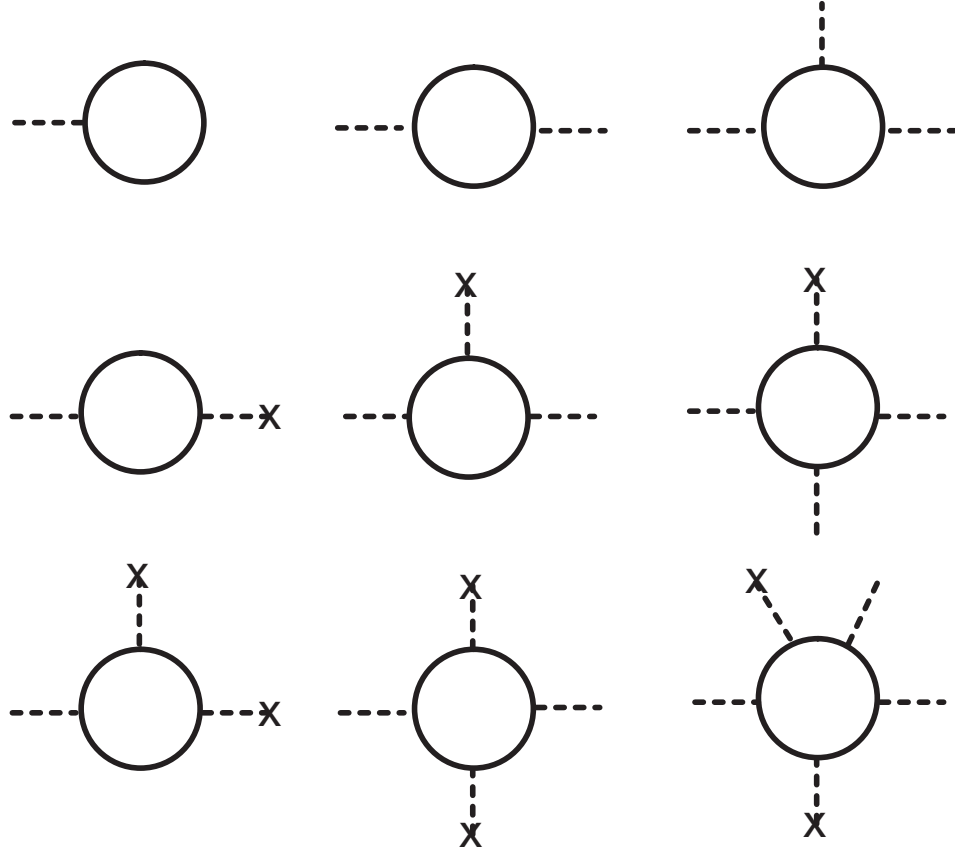


FIG. 2: Diagrams which result when the first exponential operates on the functional integral (13). Full lines represent fermion propagators, dashed lines with a cross  $\sigma \rightarrow$  vacuum transitions, dashed lines stand for either  $\sigma$  or  $\phi$  fields.

Indeed, taking into account eq. (12), one easily integrates over  $x_2$  in (13). Then, using the formula

$$\left(\frac{\delta}{\delta\Gamma(x_1)}\right)^n \exp\left(-i\int d^2x \frac{\hat{m}}{g^2}\Gamma(x)\right) = \left(-i\frac{\hat{m}}{g^2}\right)^n \exp\left(-i\int d^2x \frac{\hat{m}}{g^2}\Gamma(x)\right), \quad (14)$$

we derive for (13) an expression, where after setting  $\Gamma = 0$  and taking the first exponent out of the functional integral we get

$$\exp\left(-\hat{m}\int d^2x \frac{\delta}{\delta\sigma(x)}\right) \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-i\int d^2x [\bar{\psi}(i\cancel{\partial} + \hat{m} + \sigma + i\gamma_5\phi)\psi]\right). \quad (15)$$

Since the first exponent is a translation operator, which shifts the argument  $\sigma \rightarrow \sigma - \hat{m}$ , we obtain finally for (13)

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-i\int d^2x [\bar{\psi}(i\cancel{\partial} + \sigma + i\gamma_5\phi)\psi]\right) \propto (\det D_E|_{\hat{m}=0})^N. \quad (16)$$

Thus, we arrive at the following bare effective action

$$\text{Re } S_E^{\text{eff}} = \frac{N}{2} \text{Tr} \ln D_E^\dagger D_E|_{\hat{m}=0} - \int d^2x_E \frac{(\sigma - \hat{m})^2 + \phi^2}{2g^2}, \quad (17)$$

where the real part of the fermion determinant is invariant with respect to the chiral transformations (5)-(6), as we wished to find, and  $\delta\mathcal{L}_{\text{eff}} = \hat{m}\delta\sigma/g^2$ , as it follows from the second term.

One might think of criticizing the above calculations by claiming that the same result (that is Eq. (17)) can be obtained by the usual method, *i.e.*, by shifting  $\sigma \rightarrow \sigma - \hat{m}$  in (3) and subsequently integrating over the fermions. We argue, however, that the shift of the scalar field in the generating functional, being a replacement of the variable, changes automatically the transformation law of the pseudoscalar  $\phi$  to  $\delta\phi = -2\theta(\sigma - \hat{m})$ . As a consequence, the symmetry breaking pattern of the effective Lagrangian obtained in this way differs from (7) and, therefore, suffers from the problem indicated in Sec. II. In contrast, Eq. (17) obtained through the complementary tadpole diagrams contained in the definition (13) leaves the auxiliary fields and their transformation properties unchanged.

#### IV. TADPOLES AT THE LEVEL OF THE GENERATING FUNCTIONAL

The procedure considered above can be easily implemented in the framework of the functional integral approach. Indeed, writing the linearized Lagrangian density  $\mathcal{L}(\psi, \bar{\psi}, \sigma, \phi)$  as the sum of the symmetry breaking,  $\mathcal{L}_{\text{SB}}(\psi, \bar{\psi})$ , and symmetric,  $\mathcal{L}_{\text{S}}$ , parts, one has for the generating functional

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp i\mathcal{S}(\psi, \bar{\psi}) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{i\mathcal{S}_{\text{SB}}(\psi, \bar{\psi})} \int \mathcal{D}\sigma\mathcal{D}\phi \exp i\mathcal{S}_{\text{S}}(\psi, \bar{\psi}, \sigma, \phi), \quad (18)$$

where  $\mathcal{S}_{\text{SB}}(\psi, \bar{\psi}) = -\int d^2x \bar{\psi}\hat{m}\psi$ . At this stage, to carry out the  $1/N$  expansion, one usually integrates over the fermions assuming that the order of integration in (18) can be changed without serious consequences. This is not true, actually. The reason is that the double integral over bosonic fields is a chiral invariant, but the one over fermions is not. Thus, the pattern of symmetry breaking will be altered.

We argue now that the consistent procedure here should be as follows

$$Z = \int \mathcal{D}\sigma\mathcal{D}\phi e^{i\mathcal{S}_{\text{SB}}(\sigma)} \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp i\mathcal{S}_{\text{S}}(\psi, \bar{\psi}, \sigma, \phi), \quad (19)$$

*i.e.*, the change of the order of integrations in (18) should be accompanied with the corresponding replacement of the symmetry breaking term:  $\mathcal{L}_{\text{SB}}(\psi, \bar{\psi}) \rightarrow \mathcal{L}_{\text{SB}}(\sigma) = \hat{m}\sigma/g^2$ , where  $\mathcal{L}_{\text{SB}}(\sigma)$  is the solution of the equation  $\delta\mathcal{L}_{\text{SB}}(\sigma) = \delta\mathcal{L}_{\text{SB}}(\psi, \bar{\psi})$ . This equation can be solved, because chiral transformation properties of bosonic and fermionic fields are mutually correlated. As a result the functional integral over fermions in Eq. (19) is chirally invariant (excluding the anomaly, which is not important for the question studied here), and therefore the symmetry breaking pattern is strictly traced.

At first sight, Eq. (19) looks as if we did the shift  $\sigma \rightarrow \sigma - \hat{m}$  in (18) to obtain it. This is not the case. As we have already learned, the shift does not change the symmetry breaking pattern of the fermionic part of the Lagrangian, and therefore cannot lead to the desired chirally symmetric functional integral over fermions. Eq. (19) should instead be considered either as a postulate related with the symmetry or as the result of the  $\sigma$ -tadpole mechanism. The latter

is in our understanding the reason for the unconventional rule associated with the order of integrations in  $Z$ . This equation contains the main message of our work because it expresses the result of Sect. III in the shortest and most general form.

One can recognize in the replacement  $\mathcal{L}_{\text{SB}}(\psi, \bar{\psi}) \rightarrow \mathcal{L}_{\text{SB}}(\sigma)$  the use of the constraining relation (4). On one hand, this explains why Eq. (19) is inwardly consistent with (18). On the other hand, if one accepts that it is this field equation that controls the step from (18) to (19), the whole procedure can be straightforwardly applied to any theory with broken discrete  $\gamma_5$ -symmetry. The massive GN model [5] is an example of such a theory. In this particular case our formula leads to the result obtained by Feinberg and Zee for the effective potential.

Next, let us try to understand in simple terms what is the main difference between our calculations here and the method presented in [4]. Following those works, we would integrate out the fermions directly in Eq. (18). Of course, we would not pay much attention to the order of integrations there. However, we would calculate systematically (in the framework of the heat kernel expansion) the counterterms which are needed for consistency with the requirements of chiral symmetry and would add them to the effective Lagrangian. What would be the result? The answer is very simple. It would coincide with the effective Lagrangian which one finds integrating out the fermions in Eq. (19). To see this it is instructive to look again at Eq. (13), where the first exponent, being taken out of the integral and applied to the result of integration, generates the infinite set of counterterms which are necessary to recover the correct symmetry breaking pattern of the outcome. It is exactly this what has been done in [4], but with the use of a different and more complicated technique. It is clear that both results coincide.

## V. FINAL COMMENTS AND CONCLUSION

The renormalization of the effective Lagrangian (17) can be done similarly to the massive GN model [5]. In order that chiral symmetry constraints be valid in the renormalizable theory, it is necessary that all divergences of the theory may be absorbed in two constants, the mass  $\hat{m}$  and the coupling constant  $g^2$ . It is also important that the boson variables are chosen such that their chiral transformations do not depend on  $\hat{m}$ , otherwise one would have fatal infinities in the symmetry transformation laws. These conditions are fulfilled for (17) and we conclude that our calculations do not destroy the renormalizability of the multi-fermion Lagrangian in two space-time dimensions.

The method presented here can be easily extended to the known 4-dimensional models with nonabelian chiral symmetry and multi-fermion interactions. Several examples are given by the Nambu – Jona-Lasinio type models of QCD with explicit chiral symmetry breaking. In these cases the original quark Lagrangian satisfies the hypothesis of partially conserved axial vector current (PCAC) at the quark level and the tadpole mechanism which we suggest protects the theory from loosing this property in the large  $N_c$  limit. There are several examples in the literature where the subtle but essential difference between Eqs. (18) and (19) was not recognized, leading to an unjustified violation of the PCAC relation. Note also that the lattice calculations for the massive GN model show that PCAC is satisfied in the continuum limit [6].

To conclude, we have outlined the basic steps required to formulate the functional integration over multi-fermion interactions with explicit chiral symmetry breaking in leading  $1/N$  order. Without loss of generality we have considered the massive GN model, which is being actively studied nowadays [7]-[9]. We have shown that the method preserves the symmetry breaking pattern of the original Lagrangian and does not alter its dynamics. The information over all counterterms which are needed to be added to fulfill these requirements is cast into a very compact form and readily implemented as a rule in the functional integration.

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